Kinetic Equations

Text of the Exercises

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Exercise 1

We controlled in the proof of Theorem 8 (Lecture 6, board 3) the following quantity, defined for $n \in \mathbb{N}$, $n \geqslant 1$:

$$A(n) := 1 + 1 + 2^1 + 2^{1+2} + 2^{1+2+3} + \cdots + 2^{1+2+\cdots+(n-1)} = 1 + \sum_{k=0}^{n-1} 2^{\left(\sum_{j=0}^{k} j\right)}. \tag{1}$$

We introduced then:

$$B(n) := 2^{1+2+\cdots+n} = 2^{\frac{n(n+1)}{2}}. \tag{2}$$

Prove that $A(n) \leqslant B(n)$.

Exercise 2

We recall that the homogeneous Boltzmann equation, for radial in the velocity variable solutions f, writes:

$$\partial_t f + L(f) f = J(f), \tag{3}$$

where we saw in the last exercise sheet that L and J can be written as

$$J(f)(t,r) = 16\pi^2 \int_0^{+\infty} \int_0^{+\infty} f(t,u) f(t,v) G(r,u,v)\, uvdu\, dv, \tag{4}$$

$$G(r,u,v) = \begin{cases} 0 & \text{if } u^2+v^2 \leqslant r^2, \\ 1 & \text{if } u \geqslant r,\ v \geqslant r, \\ \frac{v}{r} & \text{if } u \geqslant r,\ v \leqslant r, \\ \frac{u}{r} & \text{if } u \leqslant r,\ v \geqslant r, \\ \frac{\sqrt{u^2+v^2-r^2}}{r} & \text{if } u^2+v^2 \geqslant r^2,\ u \leqslant r,\ v \leqslant r, \end{cases} \tag{5}$$

$$L(f)(t,r) = \int_0^{+\infty} P(r,r_1) f(t,r_1) r_1^2 dr_1, \tag{6}$$

$$P(r,r_1) = \left(2r + \frac{2r_1^2}{3r}\right) \mathbb{1}_{r_1 \leqslant r} + \left(2r_1 + \frac{2r^2}{3r_1}\right) \mathbb{1}_{r_1 > r}. \tag{7}$$

We now aim at controlling from below the solutions of (3). In order to do so, the first step is to bound from above the term $L(f)$.

We now consider a solution f of (3) on $[0,t_1] \times \mathbb{R}_+$ satisfying that there exist a constant C such that for all $(t,r) \in [0,t_1] \times \mathbb{R}_+$:

$$0 \leqslant f_0(r) \leqslant \frac{C}{(1+r)^k}, \tag{8}$$

$$0 \leqslant f(t,r) \leqslant \frac{C}{r^\alpha (1+r)^\beta}, \tag{9}$$

with $k > 6$, $0 \leqslant \alpha < 3$ and $\alpha + \beta > 6$ fixed.

Prove that there exist two positive real numbers A and B, depending only on the initial datum f_0, such that for all $r \geqslant 0$:

$$L(f)(r) \leqslant Ar + B. \tag{10}$$

Give explicitly the constants A and B.

Hint: Use Theorem 9 to control the solution f only in terms of the initial datum f_0.

Exercise 3

- Let f be a solution of (3) satisfying the assumptions of the previous exercise. Show that for all $(t, r) \in [0, t_1] \times \mathbb{R}_+$:

$$f(t,r) \geqslant f_0(r)\, e^{-(Ar+B)t} + \int_0^t e^{-(Ar+B)(t-s)} J(f(s,r))\, ds, \tag{11}$$

  where A and B are the constants found in (10).

  This is already a first interesting control on the solutions. Nevertheless, when the initial datum is 0 at r_0, the control (11) on the quantity $f(t, r_0)$ becomes trivial.

- Deduce that for all $(t, r) \in [0, t_1] \times \mathbb{R}_+$:

$$f(t,r) \geqslant 16\pi^2 t e^{-(Ar+B)t} \int_0^{+\infty} \int_0^{+\infty} G(r,u,v) \cdot \\ \cdot f_0(u)\, e^{-(Au+B)t} f_0(v)\, e^{-(Av+B)t} uv du\, dv. \tag{12}$$

- Let us now assume that the initial datum f_0 is non-zero, that is there exists $r_0 \in \mathbb{R}_+$ and $d, m > 0$ such that

$$f_0(r) \geqslant m > 0 \tag{13}$$

  for all $r \in [r_0, r_0 + d]$. Using the assumption (13), prove that

$$f(t,r) > 0 \tag{14}$$

  for all $t > 0$ and r small enough (that is, smaller than a constant which depends only on r_0 and d).

  Hint: Consider, for an arbitrary $1 < \gamma < \sqrt{2}$, the bound $b = \min_{u,v \geqslant \frac{r}{\gamma}} G(r,u,v)$, and use this quantity to bound from below the integral in (12).